

# GREEN'S FUNCTIONS IN LAYERED POROELASTIC HALF-SPACES

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## SUMMARY

In this paper, the complete Green's functions in a multilayered, isotropic, and poroelastic half-space are presented. It is the first time that all the common point sources, i.e. the total force, fluid force, fluid dilatation, and dislocation, are considered for a layered system. The Laplace transform is applied first to suppress the time variable. The cylindrical and Cartesian systems of vector functions and the propagator matrix method are then employed to derive the Green's functions. In the treatment of a point dislocation, an equivalent body-source concept is introduced, and the difference of a dislocation in a purely elastic and a poroelastic medium is discussed. While the spatial integrals involved in the Green's functions can be evaluated accurately by an adaptive Gauss quadrature with continued fraction expansions, the inverse Laplace transform can be carried out by applying a common numerical inversion technique. These complete Green's functions can be implemented into a suitable boundary element formulation to study the deformation and fracture problems in a layered poroelastic half-space. Copyright © 1999 John Wiley & Sons, Ltd.

**KEY WORDS:** Green's function; dislocation; equivalent body force; layered poroelastic half-space; vector function; propagator matrix

## 1. INTRODUCTION

Since Biot's pioneer work (References 1 and 2) on fluid-saturated porous solid, the theory of poroelasticity has been greatly developed and applied to various branches of science and engineering. While Cheng and Detournay<sup>3</sup> discussed the fundamentals of poroelasticity in details, the applications of poroelasticity were recorded in a recent book edited by Selvadurai<sup>4</sup> and a recent special issue in the *International Journal of Solids and Structures* edited by Cheng *et al.*<sup>5</sup> It is also worth mentioning the extension work by Maier and Comi<sup>6</sup> to poroplasticity, and Elsworth and Bai<sup>7</sup> to a double-porosity medium.

When analysing problems in a homogeneous, linear, and poroelastic solid, the boundary integral equation method or the boundary element method (BEM) offers significant computational advantages over the domain discretization method. When a BEM formulation is applied directly to a layered poroelastic structure, however, its merit may be lost because all the interfaces in the structure need to be discretized. An efficient way to handle this problem is to implement the Green's functions in the layered system to the BEM formulation.

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Previously, various layered Green's functions and their BEM formulations were proposed (see Reference 8 for a review). Notable studies related to poroelasticity are those by Senjuntichai and Rajapakse<sup>9</sup> and Yue and Selvadurai.<sup>10</sup> So far, however, no Green's function is available when a dislocation source is acting in a layered and poroelastic system.

This paper presents the complete Green's functions in a multilayered, isotropic, and poroelastic half-space. The common point sources, i.e. the total force, fluid force, fluid dilatation, and dislocation are all considered. While the Laplace transform is applied to suppress the time variable, the cylindrical and Cartesian systems of vector functions and the propagator matrix method are introduced to derive the Green's functions in the Laplace-transformed domain. The approach proposed here is a systematic and uniform one that can be applied to any point body-source or dislocation source (with the axially symmetric and 2-D plane sources being the special cases). In particular, when treating a dislocation, an equivalent body-force concept is introduced so that the dislocation source can be treated in the same way as for a point body-force. These Green's functions are required in the BEM modeling of deformation and fracture problems in a layered, isotropic, and poroelastic medium.

## 2. BASIC EQUATIONS OF POROELASTICITY

Biot<sup>1</sup> first introduced the theory of linear and isotropic poroelasticity for modelling the response of a fluid-saturated porous solid. The governing equations of this medium consist of the equilibrium equation and Darcy's law:

$$\sigma_{ij,j} + F_i = 0 \quad (1)$$

$$q_i = -\kappa(p_{,i} - f_i) \quad (2)$$

where  $F_i$  is the body force per unit volume acting on the mixture (fluid and solid),  $f_i$  the fluid body force,  $\kappa$  is the permeability coefficient,  $p$  the pore pressure,  $q_i$  the specific discharge, and  $\sigma_{ij}$  the total stress.

According to Rice and Cleary,<sup>11</sup> the coupled constitutive laws for the solid and fluid phases can be expressed as

$$\sigma_{ij} + bp\delta_{ij} = 2Ge_{ij} + \frac{2Gv}{1-2v}e\delta_{ij} \quad (3)$$

$$p = -\frac{2GB(1+v_u)}{3(1-2v_u)}e + \frac{2GB^2(1-2v)(1+v_u)^2}{9(v_u-v)(1-2v_u)}\varsigma \quad (4)$$

where  $e = e_{ii}$  denotes the solid volumetric strain,  $\varsigma$  is the variation of fluid volume per unit reference volume, and  $\delta_{ij}$  is the Kronecker delta,  $G$  is the shear modulus and  $B$  the Skempton's pore pressure coefficient,  $v$  and  $v_u$  are the drained and undrained Poisson's ratios, respectively.  $G$ ,  $B$ ,  $v$ ,  $v_u$ , and  $\kappa$  form a consistent set of five material parameters for the linear, isotropic, and poroelastic theory (Reference 11). The constant  $b$  in equation (3) is the Biot coefficient of effective stress (Reference 12) defined by

$$b = \frac{3(v_u - v)}{B(1-2v)(1+v_u)} \quad (5)$$

For the solid skeleton, the geometric compatibility of the strain  $e_{ij}$  with respect to the displacement  $u_i$  is assumed to be linear:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (6)$$

Finally, we need the continuity equation to complete the poroelasticity theory, which is of the form

$$\frac{\partial \varsigma}{\partial t} + q_{i,i} = \gamma \quad (7)$$

where  $\gamma$  is the rate of the injected fluid volume from a fluid source.

An alternative form of equation (4) is

$$\varsigma = \frac{\kappa(1-\nu)(1-2\nu_u)}{c(1-\nu_u)(1-2\nu)} p + be \quad (8)$$

with

$$c = \frac{2\kappa B^2 G(1-\nu)(1+\nu_u)^2}{9(1-\nu_u)(\nu_u-\nu)} \quad (9)$$

being the generalized consolidation coefficient (Reference 11). Substituting equations (2) and (8) into (7) leads to a diffusion equation for the pore pressure  $p$ , coupled by the solid dilatation  $e$ ,

$$\frac{\kappa(1-\nu)(1-2\nu_u)}{c(1-\nu_u)(1-2\nu)} \frac{\partial p}{\partial t} - (\kappa p_{,i})_{,i} + b \frac{\partial e}{\partial t} = \gamma - (\kappa f_{i,i}) \quad (10)$$

### 3. SOLUTION AND PROPAGATOR MATRICES

Although the variable decomposition technique proposed by Biot<sup>2</sup> can be employed to solve the coupled poroelastic equations (References 13 and 14), that technique is suitable only for problems in homogeneous media. Therefore, a different approach has been developed in this paper to derive the Green's functions in a layered poroelastic medium.

We first employ the Laplace transform

$$f(\mathbf{x}; s) = \int_0^{+\infty} f(\mathbf{x}; t) e^{-st} dt \quad (11)$$

to suppress the time variable  $t$  for any function depending upon time, and adopt the same symbol for the function before and after the Laplace transform. These functions are distinguished by using the Laplace variable  $s$ , in the transformed domain in the place of time  $t$ , before the transform. For instance, in the transformed domain, equation (10) becomes (we also assume that the initial value for all field quantities are zero)

$$\frac{\kappa(1-\nu)(1-2\nu_u)}{c(1-\nu_u)(1-2\nu)} sp - (\kappa p_{,i})_{,i} + bse = \gamma - (\kappa f_{i,i}) \quad (12)$$

The next step is to expand formally all field quantities in terms of the cylindrical and Cartesian systems of vector functions (Reference 15).

The cylindrical system of vector functions is defined as (Reference 15)

$$\begin{aligned}\mathbf{L}(r, \theta; \lambda, m) &= \mathbf{e}_z S(r, \theta; \lambda, m) \\ \mathbf{M}(r, \theta; \lambda, m) &= \left( \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{\partial}{r \partial \theta} \right) S(r, \theta; \lambda, m) \\ \mathbf{N}(r, \theta; \lambda, m) &= \left( \mathbf{e}_r \frac{\partial}{r \partial \theta} - \mathbf{e}_\theta \frac{\partial}{\partial r} \right) S(r, \theta; \lambda, m)\end{aligned}\quad (13)$$

with

$$S(r, \theta; \lambda, m) = \frac{1}{\sqrt{2\pi}} J_m(\lambda r) e^{im\theta} \quad (14)$$

where  $J_m(\lambda r)$  is the Bessel function of order  $m$  with  $m = 0$  corresponding to the axially symmetric deformation.

Equations (13) form an orthogonal system, and therefore any function (vector or scalar) may be expressed in terms of it. In particular, for the displacement and traction vectors, pore pressure, and specific discharge in the  $z$ -direction, we can formally write

$$\mathbf{u}(r, \theta, z; s) = \sum_m \int_0^{+\infty} [U_L(z) \mathbf{L}(r, \theta) + U_M(z) \mathbf{M}(r, \theta) + U_N(z) \mathbf{N}(r, \theta)] \lambda \, d\lambda \quad (15)$$

$$\begin{aligned}\mathbf{T}(r, \theta, z; s) &\equiv \sigma_{rz} \mathbf{e}_r + \sigma_{\theta z} \mathbf{e}_\theta + \sigma_{zz} \mathbf{e}_z \\ &= \sum_m \int_0^{+\infty} [T_L(z) \mathbf{L}(r, \theta) + T_M(z) \mathbf{M}(r, \theta) + T_N(z) \mathbf{N}(r, \theta)] \lambda \, d\lambda\end{aligned}\quad (16)$$

$$p(r, \theta, z; s) = \sum_m \int_0^{+\infty} P(z) S(r, \theta) \lambda \, d\lambda \quad (17)$$

$$q_z(r, \theta, z; s) \equiv -\kappa p_{,z} = \sum_m \int_0^{+\infty} Q(z) S(r, \theta) \lambda \, d\lambda \quad (18)$$

Notice that the dependence of the above unknown expansion coefficients on the Laplace variable  $s$  and on the parameters  $\lambda$  and  $m$  has been dropped for brevity.

Similarly, we can also expand the source functions (in the transformed domain) in equations (1) and (12) in terms of this system

$$\mathbf{F}(r, \theta, z; s) = \sum_m \int_0^{+\infty} [F_L(z) \mathbf{L}(r, \theta) + F_M(z) \mathbf{M}(r, \theta) + F_N(z) \mathbf{N}(r, \theta)] \lambda \, d\lambda \quad (19)$$

$$\gamma(r, \theta, z; s) - (\kappa f_i)_{,i}(r, \theta, z; s) = \sum_m \int_0^{+\infty} \Gamma(z) S(r, \theta) \lambda \, d\lambda \quad (20)$$

where the expansion coefficients are known for the given sources.

The definition of the Cartesian system of vector functions is given in Appendix I for the sake of completeness. It is noted, however, the following expressions of the expansion coefficients hold in these two systems. It is also emphasized that the cylindrical system of vector functions is an extension of the Hankel transform and it can be directly applied to a vector function (Reference 15).

We now substitute equations (15)–(20) into the transformed basic equations. After performing some straightforward but tedious algebra, we then obtain two independent sets of simultaneous, linear, and differential equations for the unknown coefficients involved in equations (15)–(18)

$$\begin{aligned}\frac{d[\mathbf{E}^I(z)]}{dz} &= [\mathbf{A}^I][\mathbf{E}^I(z)] + [\mathbf{V}^I(z)] \\ \frac{d[\mathbf{E}^{II}(z)]}{dz} &= [\mathbf{A}^{II}][\mathbf{E}^{II}(z)] + [\mathbf{V}^{II}(z)]\end{aligned}\quad (21)$$

where

$$\begin{aligned}[\mathbf{E}^I(z)] &= \{U_L(z), \lambda U_M(z), T_L(z)/\lambda, T_M(z), P(z), Q(z)\}^t \\ [\mathbf{E}^{II}(z)] &= \{U_N(z), T_N(z)/\lambda\}^t \\ [\mathbf{V}^I(z)] &= \{0, 0, -F_L(z)/\lambda, -F_M(z), 0, \Gamma(z)\}^t \\ [\mathbf{V}^{II}(z)] &= \{0, -F_L(z)/\lambda\}^t\end{aligned}\quad (22)$$

and  $[\mathbf{A}^I]$  and  $[\mathbf{A}^{II}]$  are the coefficient matrices with their elements being given in Appendix II.

When deriving equation (21), we have assumed that the poroelastic parameters involved are independent of the horizontal variables  $r$  and  $\theta$ , but can be any function of the vertical variable  $z$ . Although the two sets of equations can be used to obtain the transient and coupled poroelastic Green's functions in a vertically inhomogeneous half-space using the numerical integral methods (e.g. Reference 16), we assume in the following that the medium is vertically piece-wise homogeneous. This simplified model is suitable to the ground structure, and further the resulting equations can be easily solved by the propagator matrix method (Reference 17).

It is interesting that the deformation of type II (related to  $N$ ) is free of the porous effect, and thus is exactly the same as the deformation of a purely elastic solid. The author (e.g. References 15 and 18) has derived solutions associated with type-II, caused by either surface loads or point sources in a layered medium. We therefore will neglect the derivation of these solutions in the following, and give only the final results in the appropriate place for the sake of completeness.

If  $[\mathbf{A}^I]$  is a constant matrix, the homogeneous solution to equation (21) can be derived easily. Here, we employ the Laplace transform method to solve it. The advantage of using this transform is that the propagator matrix can be obtained directly from the solution matrix.

Applying the Laplace transform

$$f(\rho) = \int_0^{+\infty} f(z) e^{-\rho z} dz \quad (23)$$

to the type-I equation of (21) with  $[\mathbf{V}^I] = [\mathbf{0}]$  and inverting the result, we arrive at

$$[\mathbf{E}^I(z)] = \frac{-1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \{([\mathbf{A}^I] - \rho[\mathbf{I}])^{-1} e^{\rho z} d\rho\} [\mathbf{E}^I(0)] \quad (24)$$

where  $[\mathbf{I}]$  is the identity matrix. The inversion of the integrand matrix and the inverse Laplace integral can be derived directly, and the final result can be written simply as (also included is the type-II result)

$$\begin{aligned} [\mathbf{E}^I(z)] &= [\mathbf{a}^I(-z)][\mathbf{E}^I(0)] \\ [\mathbf{E}^{II}(z)] &= [\mathbf{a}^{II}(-z)][\mathbf{E}^{II}(0)] \end{aligned} \quad (25)$$

where  $[\mathbf{a}^I(-z)]$  and  $[\mathbf{a}^{II}(-z)]$  are the propagator matrices. By utilizing the characteristics of the propagator matrix (Reference 17), we can present equation (25) in an alternative form

$$\begin{aligned} [\mathbf{E}^I(0)] &= [\mathbf{a}^I(z)][\mathbf{E}^I(z)] \\ [\mathbf{E}^{II}(0)] &= [\mathbf{a}^{II}(z)][\mathbf{E}^{II}(z)] \end{aligned} \quad (26)$$

with the elements of  $[\mathbf{a}^I(z)]$  and  $[\mathbf{a}^{II}(z)]$  being given in Appendix III. It is noteworthy from Appendix III that while the type-II propagator matrix  $[\mathbf{a}^{II}(z)]$  is the same as that in a purely elastic solid, the type-I propagator matrix  $[\mathbf{a}^I(z)]$  can be reduced to the uncoupled purely elastic case (for indexes  $(i, j) \leq 4$ ) and purely water flow case (for indexes  $(i, j) \geq 5$ ) by setting  $b = 0$  in the matrix  $[\mathbf{a}^I(z)]$ .

#### 4. BODY-SOURCE EQUIVALENCE

When obtaining the solution and propagator matrices in the foregoing sections we have neglected the contribution from the source. If a source exists, it then causes a discontinuity on the expansion coefficients  $[\mathbf{E}^I(z)]$  and  $[\mathbf{E}^{II}(z)]$  at the source level. While the discontinuity caused by the body source (e.g.,  $F_j$  and  $\gamma - (\kappa f_i)_{,i}$ ) can be derived easily, that caused by a dislocation, however, requires special approach. Here we will first express a dislocation source by an equivalent body force (Reference 19), and then derive the discontinuities of  $[\mathbf{E}^I(z)]$  and  $[\mathbf{E}^{II}(z)]$  using the method described by Kennett.<sup>17</sup>

We start with the generalized reciprocal theorem of Betti type in poroelasticity

$$\sigma_{ij}^{(1)} e_{ij}^{(2)} + p^{(1)} \zeta^{(2)} = \sigma_{ij}^{(2)} e_{ij}^{(1)} + p^{(2)} \zeta^{(1)} \quad (27)$$

where superscripts (1) and (2) denote two independent systems of the field quantities. This relation is a direct consequence of the constitutive equations (3) and (4). Applying an instantaneous point force to the second system in equation (27) and integrating the result, we obtain the following representation relation (Reference 14):

$$\begin{aligned} u_j(\mathbf{y}, \tau) &= \iint dS(\mathbf{x}) dt \{ [\sigma_{ik}(\mathbf{x}, t) u_i^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau) - u_i(\mathbf{x}, t) \sigma_{ik}^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau)] n_k(\mathbf{x}) \\ &\quad + [v_i(\mathbf{x}, t) p^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau) - p(\mathbf{x}, t) v_i^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau)] n_i(\mathbf{x}) \} \\ &\quad + \iint dV(\mathbf{x}) dt \{ [F_i(\mathbf{x}, t) u_i^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau) + f_i(\mathbf{x}, t) v_i^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau) \\ &\quad - R(\mathbf{x}, t) p^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau)] \end{aligned} \quad (28)$$

where  $n_k$  is the outward normal at the boundary point  $\mathbf{x}$ ,

$$v_i = \int_0^t q_i dt \quad (29)$$

is the relative fluid displacement, and

$$R = \int_0^t \gamma dt \quad (30)$$

the volume of the injected fluid (i.e., the fluid dilatation). The superscript ' $Fj$ ' in equation (28) denotes the source type and its direction. For example,  $u_i^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau)$  represents the solid displacement in the  $i$ th direction at  $(\mathbf{x}, -t)$  due to an instantaneous point force  $F$  of unit impulse in the  $j$ th direction at  $(\mathbf{y}, -\tau)$ .

Let us now assume that there is an internal surface  $\Sigma$  imbedded in a homogeneous poroelastic domain  $V$  bounded externally by a surface  $S$  (Figure 1), and that  $\mathbf{n}$  is the unit normal to  $\Sigma$ . Across the surface  $\Sigma$ , the solid and fluid displacements may be discontinuous. Let  $\Delta u_i(\mathbf{x}, t)$  and  $\Delta v_i(\mathbf{x}, t)$  be the discontinuities in  $\mathbf{u}$  and  $\mathbf{v}$  across  $\Sigma$  in the  $i$ -direction at  $(\mathbf{x}, t)$ . These discontinuities may have any form, provided that the following relations hold:

$$\begin{aligned} \sigma_{ij}^+ n_j^+ + \sigma_{ij}^- n_j^- &= 0 \\ p^+ - p^- &= 0 \end{aligned} \quad (31)$$

where  $-n_j^+ = n_j^- = n_j$ .

Assuming  $v_i$ ,  $u_i$ ,  $p$  and  $\sigma_{ik}$  satisfy the same homogeneous boundary conditions on  $S$ , and applying equation (28) to the region bounded internally by  $\Sigma$  and externally by  $S$ , we arrive at

$$\begin{aligned} u_j(\mathbf{y}, \tau) &= \iint d\Sigma(\eta) dt \{ \Delta u_i(\eta, t) \sigma_{ik}^{Fj}(\eta, -t; \mathbf{y}, -\tau) n_k(\eta) - \Delta v_i(\eta, t) p^{Fj}(\eta, -t; \mathbf{y}, -\tau) n_i(\eta) \} \\ &+ \iiint dV(\mathbf{x}) dt \{ F_i(\mathbf{x}, t) u_i^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau) + f_i(\mathbf{x}, t) v_i^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau) \\ &- R(\mathbf{x}, t) p^{Fj}(\mathbf{x}, -t; \mathbf{y}, -\tau) \} \end{aligned} \quad (32)$$

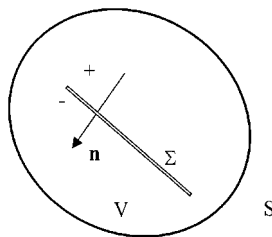


Figure 1. An internal surface  $\Sigma$  imbedded in a poroelastic domain  $V$  bounded externally by the surface  $S$ .  $\mathbf{n}$  is the unit normal vector on  $\Sigma$

Making change of the variables in the point force solution (Reference 20), equation (32) then becomes

$$u_j(\mathbf{y}, \tau) = \iint d\Sigma(\eta) dt \{ -\Delta u_i(\eta, t) \sigma_{ik}^{Fj}(\mathbf{y}, \tau; \eta, t) n_k(\eta) + \Delta v_i(\eta, t) p^{Fj}(\mathbf{y}, \tau; \eta, t) n_i(\eta) \} \\ + \iint dV(\mathbf{x}) dt \{ F_i(\mathbf{x}, t) u_i^{Fj}(\mathbf{y}, \tau; \mathbf{x}, t) + f_i(\mathbf{x}, t) v_i^{Fj}(\mathbf{y}, \tau; \mathbf{x}, t) + R(\mathbf{x}, t) p^{Fj}(\mathbf{y}, \tau; \mathbf{x}, t) \} \quad (33)$$

We now derive the body-force equivalence of the solid and fluid dislocations. Using the following identity:

$$p^{Fj}(\mathbf{y}, \tau; \eta, t) = \int dV(\mathbf{x}) \delta(\mathbf{x}, \eta) p^{Fj}(\mathbf{y}, \tau; \mathbf{x}, t) \quad (34)$$

the second surface integral in equation (33) can be expressed alternatively as

$$\iint d\Sigma(\eta) dt \{ \Delta v_i(\eta, t) p^{Fj}(\mathbf{y}, \tau; \eta, t) n_i(\eta) \} \\ + \iint dV(\mathbf{x}) dt \left\{ \left[ \iint d\Sigma(\eta) \Delta v_i(\eta, t) n_i(\eta) \delta(\mathbf{x}, \eta) \right] p^{Fj}(\mathbf{y}, \tau; \mathbf{x}, t) \right\} \quad (35)$$

Comparison of this expression to the last volume integral in equation (33), we conclude that the effect of a fluid dislocation  $\Delta v_i n_i$  (the second surface integral in equation (33)) is equivalent to that of a fluid dilatation  $\Gamma$ . Furthermore, the body-source equivalence of a fluid dislocation  $\Delta v_i n_i$  is equal to

$$R^{ef}(\mathbf{x}, t) = \int d\Sigma(\eta) \Delta v_i(\eta, t) n_i(\eta) \delta(\mathbf{x}, \eta) \quad (36)$$

The derivation of the body-source equivalence of the first surface integral in equation (33) is somewhat difficult. We start with the stresses caused by a point force, which can be expressed as

$$\sigma_{ik}^{Fj}(\mathbf{y}, \tau; \eta, t) = c_{iklm} u_{l,m}^{Fj}(\mathbf{y}, \tau; \eta, t) - b \delta_{ik} p^{Fj}(\mathbf{y}, \tau; \eta, t) \quad (37)$$

where

$$c_{iklm} = \frac{2G\nu}{1-2\nu} \delta_{ik} \delta_{lm} + G(\delta_{il} \delta_{km} + \delta_{im} \delta_{kl}) \quad (38)$$

Substituting equation (37) into (33), the first surface integral of equation (33) then becomes

$$\iint d\Sigma(\eta) dt \{ -\Delta u_i(\eta, t) \sigma_{ik}^{Fj}(\mathbf{y}, \tau; \eta, t) n_k(\eta) \} \\ = \iint d\Sigma(\eta) dt \{ -\Delta u_i(\eta, t) c_{iklm} u_{l,m}^{Fj}(\mathbf{y}, \tau; \eta, t) n_k(\eta) \\ + b \Delta u_i(\eta, t) n_i(\eta) p^{Fj}(\mathbf{y}, \tau; \eta, t) \} \quad (39)$$



It is obvious from this expression that a solid dislocation is equivalent to a summation of two body sources, with the second one being a fluid dilatation. The equivalence of the second part is

$$R^{\text{ed}}(\mathbf{x}, t) = b \int d\Sigma(\eta) \Delta u_i(\eta, t) n_i(\eta) \delta(\mathbf{x}, \eta) \quad (40)$$

In order to find the equivalence of the first part on the right-hand side of equation (39), we use the relation

$$\begin{aligned} u_{l,m}^{Fj}(\mathbf{y}, \tau; \eta, t) &= \left( \frac{\partial}{\partial y_m} \right) u_l^{Fj}(\mathbf{y}, \tau; \eta, t) = - \left( \frac{\partial}{\partial \eta_m} \right) u_l^{Fj}(\mathbf{y}, \tau; \eta, t) \\ &= \int dV(\mathbf{x}) \frac{\partial \delta(\mathbf{x}, \eta)}{\partial x_m} u_l^{Fj}(\mathbf{y}, \tau; \mathbf{x}, t) \end{aligned} \quad (41)$$

Therefore, the first part can be expressed as

$$\begin{aligned} &\iint d\Sigma(\eta) dt \{ -\Delta u_i(\eta, t) c_{iklm} u_{l,m}^{Fj}(\mathbf{y}, \tau; \eta, t) n_k(\eta) \} \\ &= \iint dV(\mathbf{x}) dt \left\{ - \int d\Sigma(\eta) \left[ c_{iklm} \Delta u_i(\eta, t) n_k(\eta) \frac{\partial \delta(\mathbf{x}, \eta)}{\partial x_m} \right] u_l^{Fj}(\mathbf{y}, \tau; \mathbf{x}, t) \right\} \end{aligned} \quad (42)$$

Comparison of this expression to the first volume integral in equation (33), we find that this part of the solid dislocation is equivalent to a body force, with its equivalence being

$$F_l^{\text{ed}}(\mathbf{x}, t) = - \int d\Sigma(\eta) c_{iklm} \Delta u_i(\eta, t) n_k(\eta) \frac{\partial \delta(\mathbf{x}, \eta)}{\partial x_m} \quad (43)$$

We conclude that while a fluid dislocation is equivalent to a fluid dilatation, which is given by equation (36), the body-source equivalence of a solid dislocation is the sum of a fluid dilatation and a body force, i.e. equations (40) and (43). This result makes the ‘equivalence’ concept in poroelasticity different to that in the pure elasticity. For an isotropic and purely elastic medium, it has been shown that the response caused by a dislocation can be generated by certain combination of double forces (References 19 and 21). In a poroelastic medium, however, a fluid dilatation as well as the double forces must be used to generate the response caused by a solid dislocation. Furthermore, we need an equivalent fluid dilatation to generate the response due to a fluid dislocation.

## 5. SOURCE FUNCTIONS

In Section 3, we derived the homogeneous solutions and the associated propagator matrices in a poroelastic and layered half-space. When a point source exists, however, the problem becomes quite complicated. In order to derive the response of a layered half-space to a point source, i.e. the Green’s function, we need first to obtain the source function, or the discontinuity of the expansion coefficients of the field quantities caused by the point source. This can be achieved by expanding

the point source in terms of the cylindrical and Cartesian systems of vector functions (References 15 and 18), substituting the expansion coefficients into equation (21), and deriving the solution to this equation. While in this section, we address all the source functions, the Green's functions corresponding to these source functions will be derived in the next section.

### 5.1. Solid point force

Without loss of generality, we assume that there is a solid point force located along the  $z$ -axis at the depth  $z = h$  (also for other point sources), and that in the Laplace transformed domain it can be expressed as

$$F_j(r, \theta, z; s) = F(s) \frac{\delta(r) \delta(\theta) \delta(z - h)}{r} n_j, \quad j = r, \theta, z \quad (44)$$

where  $F(s)$  is the Laplace transform of the time amplitude factor,  $(n_r, n_\theta, n_z)$  are the direction cosines of the unit force vector in the cylindrical coordinates  $(r, \theta, z)$ . It can be shown that this point force will cause the following discontinuity on the expansion coefficients of the traction vector:

$$\begin{aligned} \Delta T_L &\equiv T_L(h+0) - T_L(h-0) = F(s) \frac{-n_z}{\sqrt{2\pi}}, \quad m = 0 \\ \Delta T_M &\equiv T_M(h+0) - T_M(h-0) = F(s) \frac{\mp n_x + i n_y}{2\lambda \sqrt{2\pi}}, \quad m = \pm 1 \\ \Delta T_N &\equiv T_N(h+0) - T_N(h-0) = F(s) \frac{i n_x \pm n_y}{2\lambda \sqrt{2\pi}}, \quad m = \pm 1 \end{aligned} \quad (45)$$

where  $i = \sqrt{-1}$ ,  $(n_x, n_y, n_z)$  are the  $(x, y, z)$  direction cosines of the unit force vector in the space fixed Cartesian coordinates, with  $x$ - and  $y$ -directions being taken, respectively, along  $\theta = 0$  and  $\theta = \pi/2$  of the cylindrical coordinates.

A parallel result in the Cartesian system of vector functions was found to be (Reference 22)

$$\begin{aligned} \Delta T_L &\equiv T_L(h+0) - T_L(h-0) = -F(s) \frac{n_z}{2\pi} \\ \Delta T_M &\equiv T_M(h+0) - T_M(h-0) = -iF(s) \frac{n_x \alpha + n_y \beta}{2\pi \lambda^2} \\ \Delta T_N &\equiv T_N(h+0) - T_N(h-0) = -iF(s) \frac{n_x \beta - n_y \alpha}{2\pi \lambda^2} \end{aligned} \quad (46)$$

### 5.2. Fluid point source

We assume that a fluid point source in its Laplace transform can be expressed as

$$\gamma(r, \theta, z; s) - (\kappa f_i)_{,i}(r, \theta, z; s) = D(s) \frac{\delta(r) \delta(\theta) \delta(z - h)}{r} \quad (47)$$

with  $D(s)$  being the Laplace transformed amplitude of the fluid source. Similar to the above procedure, we find that the corresponding source functions in the cylindrical and Cartesian systems of vector functions are, respectively,

$$Q(h+0) - Q(h-0) = \frac{D(s)}{\sqrt{2\pi}}, \quad m=0 \quad (48)$$

$$Q(h+0) - Q(h-0) = \frac{D(s)}{2\pi} \quad (49)$$

### 5.3. Fluid point dislocation action on $d\Sigma$

For a fluid point dislocation, we obtain, from equation (36), the equivalence of its fluid dilatation in the Laplace transformed domain

$$R(r, \theta, z; s) = \Delta v(s) v_i n_i \frac{\delta(r) \delta(\theta) \delta(z-h)}{r} \quad (50)$$

where  $v_i$  and  $n_i$  are the dislocation direction and the normal to the dislocation surface, respectively, and  $\Delta v(s)$  is the Laplace transform of the dislocation amplitude  $\Delta v(t)$  ( $\Delta v_i = \Delta v v_i$ ).

Utilizing relation (30) and the results in Section 5.2, we find that the source functions due to a fluid point dislocation in the cylindrical and Cartesian systems of vector functions are, respectively,

$$Q(h+0) - Q(h-0) = \Delta v(s) s \frac{n_x v_x + n_y v_y + n_z v_z}{\sqrt{2\pi}}, \quad m=0 \quad (51)$$

$$Q(h+0) - Q(h-0) = \Delta v(s) s \frac{n_x v_x + n_y v_y + n_z v_z}{2\pi} \quad (52)$$

### 5.4. Solid point dislocation acting on $d\Sigma$

We have shown in Section 4 that a solid dislocation produces two components of the body-source equivalence: one equivalent to a fluid dilatation and another to a solid force. This equivalence has just been derived, as given by equations (40) and (43). Therefore, in order to obtain the response from an arbitrary solid dislocation, we need only to sum up the responses from its body-source equivalence (40) and (43).

The first equivalence is similar to a fluid point dislocation, and thus its source functions are similar to equations (51) and (52). That is, in the cylindrical and Cartesian systems of vector functions, they are respectively,

$$Q(h+0) - Q(h-0) = \Delta u(s) s b \frac{n_x v_x + n_y v_y + n_z v_z}{\sqrt{2\pi}}, \quad m=0 \quad (53)$$

$$Q(h+0) - Q(h-0) = \Delta u(s) s b \frac{n_x v_x + n_y v_y + n_z v_z}{2\pi} \quad (54)$$

where  $\Delta u(s)$  is the Laplace transform of the solid dislocation amplitude  $\Delta u(t)$  related to its component as

$$\Delta u_i(t) = \Delta u(t)v_i \quad (55)$$

The source functions from the second equivalence (43), however, are somewhat complicated to derive. Fortunately, for a transversely isotropic and purely elastic solid, the author (Reference 18) has obtained the results. Therefore, we can directly reduce them to the isotropic case. In the cylindrical system of vector functions, they are (omitting the factor  $\Delta u(s)/\sqrt{2\pi}$ )

$$\begin{aligned} U_L(h+0) - U_L(h-0) &= \frac{v}{1-v} (n_x v_x + n_y v_y) + n_z v_z, \quad m = 0 \\ U_M(h+0) - U_M(h-0) &= [\pm (n_x v_z + n_z v_x) - i(n_y v_z + n_z v_y)]/(2\lambda), \quad m = \pm 1 \\ U_N(h+0) - U_N(h-0) &= [-i(n_x v_z + n_z v_x) \mp (n_y v_z + n_z v_y)]/(2\lambda), \quad m = \pm 1 \\ T_M(h+0) - T_M(h-0) &= \frac{G(1+v)}{1-v} (n_x v_x + n_y v_y), \quad m = 0 \\ &+ 0.5G[(n_y v_y - n_x v_x) \pm i(n_x v_y + n_y v_x)], \quad m = \pm 2 \\ T_N(h+0) - T_N(h-0) &= 0.5G[(n_x v_y + n_y v_x) \pm i(n_x v_x - n_y v_y)], \quad m = \pm 2 \\ Q(h+0) - Q(h-0) &= -bs \left[ \frac{v}{1-v} (n_x v_x + n_y v_y) + n_z v_z \right], \quad m = 0 \end{aligned} \quad (56)$$

In the Cartesian system of vector functions, they are (again, omitting the factor  $\Delta u(s)/(2\pi)$ ).

$$\begin{aligned} U_L(h+0) - U_L(h-0) &= \frac{v}{1-v} (n_x v_x + n_y v_y) + n_z v_z \\ U_M(h+0) - U_M(h-0) &= i[\alpha(n_x v_z + n_z v_x) + \beta(n_y v_z + n_z v_y)]/\lambda^2 \\ U_N(h+0) - U_N(h-0) &= i[\beta(n_x v_z + n_z v_x) - \alpha(n_y v_z + n_z v_y)]/\lambda^2 \\ T_M(h+0) - T_M(h-0) &= \frac{2Gv^2}{(1-2v)(1-v)} (n_x v_x + n_y v_y) \\ &+ \frac{2G}{\lambda^2} \left[ \frac{(1-v)\alpha^2 + v\beta^2}{(1-2v)} n_x v_x \right. \\ &\left. + \frac{(1-v)\beta^2 + v\alpha^2}{(1-2v)} n_y v_y + \alpha\beta(n_x v_y + n_y v_x) \right] \\ T_N(h+0) - T_N(h-0) &= G[(n_x v_y + n_y v_x)(\beta^2 - \alpha^2) + 2\alpha\beta(n_x v_x - n_y v_y)]/\lambda^2 \\ Q(h+0) - Q(h-0) &= -bs \left[ \frac{v}{1-v} (n_x v_x + n_y v_y) + n_z v_z \right] \end{aligned} \quad (57)$$

Comparison of these source functions to those in a purely elastic medium (Reference 18), we find that in poroelasticity, a term associate with a fluid dilatation need to be added to the source functions in order to obtain an exact equivalence between a solid dislocation and a body source. By summing up the two source functions for  $Q$ , we have the total discontinuity of  $Q$  in the cylindrical and Cartesian systems of vector functions, respectively (again, the factors  $\Delta u(s)/\sqrt{2\pi}$  and  $\Delta u(s)/(2\pi)$  are omitted, respectively),

$$Q(h+0) - Q(h-0) = bs \frac{1-2\nu}{1-\nu} (n_x v_x + n_y v_y), \quad m=0 \quad (58)$$

$$Q(h+0) - Q(h-0) = bs \frac{1-2\nu}{1-\nu} (n_x v_x + n_y v_y) \quad (59)$$

The source functions in two-dimensional  $(y, z)$  plane can be obtained directly from the above expressions in the Cartesian system by replacing  $2\pi$  and  $\alpha$  with  $\sqrt{2\pi}$  and 0, respectively (Reference 18). In addition, the dislocation type  $n_x v_x$  should be omitted since it does not occur in a  $(y, z)$  plane problem.

## 6. POROELASTIC SOLUTIONS FOR LAYERED HALF-SPACES

The propagator matrices and various source functions obtained above are now used to derive the Green's functions in a poroelastic and layered half-space. Figure 2 is a schematic layered system, which consists of  $p-1$  poroelastic and homogeneous layers lying over a homogeneous and poroelastic half-space. We number the layer serially, with the layer at the top being layer 1 and the half-space layer  $p$ . The origins of the cylindrical and Cartesian coordinates are placed at the surface, and the  $z$ -axis is drawn into the medium. The  $k$ th layer has the thickness of  $h_k$ , and is bounded by the interfaces  $z = z_{k-1}, z_k$ . It is obvious that  $z_0 = 0$  and  $z_{p-1} = H$ . Furthermore, we assume that appropriate boundary conditions (in particular, traction-free and fluid flux-free) are applied to the top surface  $z = 0$ , and that a point source is located on the  $z$ -axis at a depth  $h$  below the surface. Let the source layer be designated as layer  $s$  with boundaries  $z = z_{s-1}, z_s$ . We divide this layer into two sub-layers,  $s1$  and  $s2$ , of identical properties. The first sub-layer is bounded by the planes  $z = z_{s-1}, h$ , and the second by  $z = h, z_s$ . Finally, displacement and traction vectors,

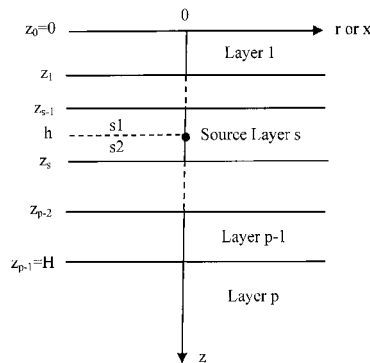


Figure 2. A layered poroelastic half-space with a point source acting along the  $z$ -axis at a depth  $z = h$

pore pressure and the specific discharge in the  $z$ -direction are assumed to be continuous across any interfaces of layers except at  $z = h$  (Reference 23). It can be shown that the continuity of these quantities is equivalent to the continuity of the column matrices  $[\mathbf{E}^I(z)]$  and  $[\mathbf{E}^{II}(z)]$ . Across the source level  $z = h$ , however,  $[\mathbf{E}^I(z)]$  and  $[\mathbf{E}^{II}(z)]$  will experience a jump or discontinuity. These discontinuities have been just discussed in the previous section for various sources, and in general, they can be expressed as

$$\begin{aligned} [\Delta \mathbf{E}^I] &= [\mathbf{E}_{s2}^I(h)] - [\mathbf{E}_{s1}^I(h)] \\ [\Delta \mathbf{E}^{II}] &= [\mathbf{E}_{s2}^{II}(h)] - [\mathbf{E}_{s1}^{II}(h)] \end{aligned} \quad (60)$$

We now apply the propagating relation (26) to the  $k$ th layer

$$\begin{aligned} [\mathbf{E}^I(z_{k-1})] &= [\mathbf{a}_k^I(z_k - z_{k-1})][\mathbf{E}^I(z_k)] \\ [\mathbf{E}^{II}(z_{k-1})] &= [\mathbf{a}_k^{II}(z_k - z_{k-1})][\mathbf{E}^{II}(z_k)] \end{aligned} \quad (61)$$

This relation makes a connection of the quantities at the upper and lower interfaces of layer  $k$ . Propagating this relation from the top of the source  $z = h - 0$  to the surface  $z = 0$ , we have

$$\begin{aligned} [\mathbf{E}^I(0)] &= [\mathbf{a}_1^I][\mathbf{a}_2^I] - -[\mathbf{a}_{s1}^I][\mathbf{E}_{s1}^I(h)] \\ [\mathbf{E}^{II}(0)] &= [\mathbf{a}_1^{II}][\mathbf{a}_2^{II}] - -[\mathbf{a}_{s1}^{II}][\mathbf{E}_{s1}^{II}(h)] \end{aligned} \quad (62)$$

Similarly, propagating from the half-space  $z = H$  to the bottom of the source  $z = h + 0$ , we arrive at

$$\begin{aligned} [\mathbf{E}_{s2}^I(h)] &= [\mathbf{a}_{s2}^I][\mathbf{a}_{s+1}^I] - -[\mathbf{a}_{p-1}^I][\mathbf{Z}_p^I(H)][\mathbf{K}^I] \\ [\mathbf{E}_{s2}^{II}(h)] &= [\mathbf{a}_{s2}^{II}][\mathbf{a}_{s+1}^{II}] - -[\mathbf{a}_{p-1}^{II}][\mathbf{Z}_p^{II}(H)][\mathbf{K}^{II}] \end{aligned} \quad (63)$$

where  $[\mathbf{Z}_p^I(H)]$  and  $[\mathbf{Z}_p^{II}(H)]$  are the homogeneous solution matrices evaluated at  $z = H$ , with their elements being given in Appendix IV. Also in equation (63),  $[\mathbf{K}^I]$  and  $[\mathbf{K}^{II}]$  are constant column matrices with components

$$\begin{aligned} [\mathbf{K}^I] &= \{0, c_2, 0, c_4, 0, c_6\}^t \\ [\mathbf{K}^{II}] &= \{0, c_8\}^t \end{aligned} \quad (64)$$

From equations (62) and (63), we find

$$\begin{aligned} [\mathbf{E}^I(0)] &= [\mathbf{D}^I][\mathbf{K}^I] - [\mathbf{W}^I] \\ [\mathbf{E}^{II}(0)] &= [\mathbf{D}^{II}][\mathbf{K}^{II}] - [\mathbf{W}^{II}] \end{aligned} \quad (65)$$

where

$$\begin{aligned} [\mathbf{D}^I] &= [\mathbf{a}_1^I][\mathbf{a}_2^I] - -[\mathbf{a}_{p-1}^I][\mathbf{Z}_p^I(H)] \\ [\mathbf{D}^{II}] &= [\mathbf{a}_1^{II}][\mathbf{a}_2^{II}] - -[\mathbf{a}_{p-1}^{II}][\mathbf{Z}_p^{II}(H)] \\ [\mathbf{W}^I] &= [\mathbf{a}_1^I][\mathbf{a}_2^I] - -[\mathbf{a}_{s1}^I][\Delta \mathbf{E}^I] \\ [\mathbf{W}^{II}] &= [\mathbf{a}_1^{II}][\mathbf{a}_2^{II}] - -[\mathbf{a}_{s1}^{II}][\Delta \mathbf{E}^{II}] \end{aligned} \quad (66)$$

Using the free boundary conditions at  $z = 0$ , i.e.

$$T_L(0) = T_M(0) = T_N(0) = Q(0) = 0 \quad (67)$$

the constants in  $[\mathbf{K}^I]$  and  $[\mathbf{K}^II]$  can be determined as

$$c_8 = W_2^{II}/D_{22}^{II}$$

$$c_2 = \frac{1}{\Delta} \begin{bmatrix} W_3^I & D_{34}^I & D_{36}^I \\ W_4^I & D_{44}^I & D_{46}^I \\ W_6^I & D_{64}^I & D_{66}^I \end{bmatrix}, \quad c_4 = \frac{1}{\Delta} \begin{bmatrix} D_{32}^I & W_3^I & D_{36}^I \\ D_{42}^I & W_4^I & D_{46}^I \\ D_{62}^I & W_6^I & D_{66}^I \end{bmatrix}, \quad c_6 = \frac{1}{\Delta} \begin{bmatrix} D_{32}^I & D_{34}^I & W_3^I \\ D_{42}^I & D_{44}^I & W_4^I \\ D_{62}^I & D_{64}^I & W_6^I \end{bmatrix}, \quad (68)$$

where

$$\Delta = \begin{bmatrix} D_{32}^I & D_{34}^I & D_{36}^I \\ D_{42}^I & D_{44}^I & D_{46}^I \\ D_{62}^I & D_{64}^I & D_{66}^I \end{bmatrix}, \quad (69)$$

Once the constants  $c_i$  in equation (64) are determined, the Green's functions in any vertical level can be obtained. For any point below the source level, i.e.  $z \geq h + 0$  (suppose that  $z$  is in layer  $k$ , i.e.  $z_{k-1} \leq z \leq z_k$ ), we have

$$\begin{aligned} [\mathbf{E}^I(z)] &= [\mathbf{a}_k^I(z_k - z)][\mathbf{a}_{k+1}^I] - -[\mathbf{a}_{p-1}^I][\mathbf{Z}_p^I(H)][\mathbf{K}^I] \\ [\mathbf{E}^{II}(z)] &= [\mathbf{a}_k^{II}(z_k - z)][\mathbf{a}_{k+1}^{II}] - -[\mathbf{a}_{p-1}^{II}][\mathbf{Z}_p^{II}(H)][\mathbf{K}^{II}] \end{aligned} \quad (70)$$

Similarly, for any point above the source, i.e.  $z \leq h - 0$  (suppose that  $z$  is in layer  $j$ , i.e.  $z_{j-1} \leq z \leq z_j$ ), we arrive at

$$\begin{aligned} [\mathbf{E}^I(z)] &= [\mathbf{a}_j^I(z_j - z)][\mathbf{a}_{j+1}^I] - -[\mathbf{a}_{p-1}^I][\mathbf{Z}_p^I(H)][\mathbf{K}^I] - [\mathbf{a}_j^I(z_j - z)][\mathbf{a}_{j+1}^I] - -[\mathbf{a}_{s1}^I][\Delta \mathbf{E}^I] \\ [\mathbf{E}^{II}(z)] &= [\mathbf{a}_j^{II}(z_j - z)][\mathbf{a}_{j+1}^{II}] - -[\mathbf{a}_{p-1}^{II}][\mathbf{Z}_p^{II}(H)][\mathbf{K}^{II}] - [\mathbf{a}_j^{II}(z_j - z)][\mathbf{a}_{j+1}^{II}] - -[\mathbf{a}_{s1}^{II}][\Delta \mathbf{E}^{II}] \end{aligned} \quad (71)$$

Now, we can substitute these expansion coefficients  $[\mathbf{E}^I]$  and  $[\mathbf{E}^{II}]$  into equations (15)–(18) to find the corresponding field quantities in the Laplace transformed domain. The remaining stress and specific discharge components are shown to be a linear combination of the known coefficients  $[\mathbf{E}^I]$  and  $[\mathbf{E}^{II}]$ , and their expressions in the Laplace transformed domain are given in Appendix V in terms of both the cylindrical and Cartesian systems of vector functions. The time-domain solution can be obtained by a numerical inverse Laplace transform.

## 7. CONCLUSIONS

The complete Green's solutions have been derived in this paper for a multilayered, isotropic, and poroelastic half-space. These Green's functions are expressed in terms of the inverse Laplace

transform (for time  $t$ ) and of the cylindrical and Cartesian systems of vector functions (for the horizontal variables  $r, \theta$  or  $x, y$ ). The advantage of using these systems of vector functions is that the solution and propagator matrices have the same structures in these two systems, and that the axially symmetric and 2-D plane deformations are the special cases of the current complete solutions.

The propagator matrix method has also been introduced to avoid solving a large system of equations with order proportional to the layer number. It is also the first time that a dislocation in a layered poroelastic system has been considered in details by an equivalent body-source concept. While the Green's functions due to the body source can be incorporated into a BEM formulation for the deformation and stress analysis, those due to the fluid and solid dislocations are required in the BEM modelling of hydrofracture problems in a layered poroelastic half-space. Numerical implementation of those Green's functions into a suitable BEM formulation (Reference 24) is currently under investigation by the author and progress will be reported in the future.

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### NOTATION

$G$	shear modulus
$B$	Skempton's pore pressure coefficient
$\nu$	drained Poisson's ratio
$\nu_u$	undrained Poisson's ratio
$\kappa$	permeability coefficient
$b$	Biot's coefficient of effective stress
$c$	generalized consolidation coefficient
$\sigma_{ij}$	total stress tensor
$F_i$	bulk body force (on fluid and solid)
$f_i$	fluid body force
$p$	pore pressure
$\gamma$	rate of injected fluid volume
$R = \int_0^t \gamma dt$	fluid dilatation (volume of injected fluid)
$u_i$	solid displacement
$\Delta u_i$	discontinuity of solid displacement (or solid dislocation)
$e_{ij}$	strain tensor
$e = e_{ii}$	volumetric strain (or solid dilatation)
$\varsigma$	variation of fluid volume
$q_i$	specific discharge
$v_i = \int_0^t q_i dt$	relative fluid displacement
$\Delta v_i$	discontinuity of relative fluid displacement (or fluid dislocation)
$t, \tau$	time variable
$s$	Laplace variable; also used to denote the source layer number



$S$	scalar function; also used to denote the external boundary
$\mathbf{L}, \mathbf{M}, \mathbf{N}$	vector functions
$U_L, U_M, U_N$	expansion coefficients of solid displacement $u_i$
$T_L, T_M, T_N$	expansion coefficients of traction $t_i = \sigma_{iz}$
$P$	expansion coefficient of pore pressure
$Q$	expansion coefficient of specific discharge $q_z$
$F_L, F_M, F_N$	expansion coefficients of body force $F_i$
$\Gamma$	expansion coefficient of the fluid source $\gamma - (\kappa f_i)_{,i}$
$D$	Laplace transformed amplitude of the fluid source $\gamma - (\kappa f_i)_{,i}$
$n_i$	normal cosines to a surface
$v_i$	direction cosines of a dislocation or displacement discontinuity
$\Sigma$	discontinuous surface
$\eta$	variable vector on $\Sigma$

## APPENDIX I

*Cartesian system of vector functions*

The Cartesian system of vector functions is defined as (Reference 15)

$$\begin{aligned}\mathbf{L}(x, y; \alpha, \beta) &= \mathbf{e}_z S(x, y; \alpha, \beta) \\ \mathbf{M}(x, y; \alpha, \beta) &= \left( \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} \right) S(x, y; \alpha, \beta) \\ \mathbf{N}(x, y; \alpha, \beta) &= \left( \mathbf{e}_x \frac{\partial}{\partial y} - \mathbf{e}_y \frac{\partial}{\partial x} \right) S(x, y; \alpha, \beta)\end{aligned}\quad (72)$$

with

$$S(x, y; \alpha, \beta) = \frac{1}{2\pi} e^{-i(\alpha x + \beta y)} \quad (73)$$

where  $i = \sqrt{-1}$ .

Equation (72) forms an orthogonal system and any function (vector or scalar) may be expressed in terms of it. In particular, for the displacement and traction vectors, pore pressure, and specific discharge in the  $z$ -direction, we have formally

$$\mathbf{u}(x, y, z; s) = \iint_{-\infty}^{+\infty} [U_L(z)\mathbf{L}(x, y) + U_M(z)\mathbf{M}(x, y) + U_N(z)\mathbf{N}(x, y)] d\alpha d\beta \quad (74)$$

$$\begin{aligned}\mathbf{T}(x, y, z; s) &\equiv \sigma_{xz}\mathbf{e}_x + \sigma_{yz}\mathbf{e}_y + \sigma_{zz}\mathbf{e}_z \\ &= \iint_{-\infty}^{+\infty} [T_L(z)\mathbf{L}(x, y) + T_M(z)\mathbf{M}(x, y) + T_N(z)\mathbf{N}(x, y)] d\alpha d\beta\end{aligned}\quad (75)$$

$$p(x, y, z; s) = \iint_{-\infty}^{+\infty} P(z)S(x, y) d\alpha d\beta \quad (76)$$

$$q_z(x, y, z; s) \equiv -\kappa p_{,z} = \iint_{-\infty}^{+\infty} Q(z)S(x, y) d\alpha d\beta \quad (77)$$

As in the text, the dependence of the above unknown expansion coefficients on the Laplace variable  $s$  and on the parameters  $\alpha$  and  $\beta$  has been dropped for brevity.

Similarly, we can also expand the source functions in equations (1) and (12) in terms of this system

$$\mathbf{F}(x, y, z; s) = \int \int_{-\infty}^{+\infty} [F_L(z)\mathbf{L}(x, y) + F_M(z)\mathbf{M}(x, y) + F_N(z)\mathbf{N}(x, y)] d\alpha d\beta \quad (78)$$

$$\gamma(x, y, z; s) - (\kappa f_i)_{,i}(x, y, z; s) = \int \int_{-\infty}^{+\infty} \Gamma(z)S(x, y) d\alpha d\beta \quad (79)$$

where the expansion coefficients are known for the given sources. It is noteworthy that the coefficient, solution, and propagator matrices given in the text for the cylindrical system hold also for the Cartesian system of vector functions with  $\lambda$  being recognized as  $\sqrt{(\alpha^2 + \beta^2)}$ .

## APPENDIX II

*Coefficient matrices  $[\mathbf{A}^I]$  and  $[\mathbf{A}^{II}]$*

The coefficient matrix  $[\mathbf{A}^I]$  in equation (21) is equal to

$$\begin{bmatrix} 0 & \frac{\lambda v}{1-v} & \frac{\lambda(1-2v)}{2G(1-v)} & 0 & \frac{b(1-2v)}{2G(1-v)} & 0 \\ -\lambda & 0 & 0 & \frac{\lambda}{G} & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & \frac{2G\lambda}{1-v} & \frac{-\lambda v}{1-v} & 0 & \frac{b(1-2v)}{1-v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{\kappa} \\ 0 & \frac{\lambda bs(1-2v)}{1-v} & \frac{-\lambda bs(1-2v)}{2G(1-v)} & 0 & -a & 0 \end{bmatrix} \quad (80)$$

where

$$a = \kappa(\lambda^2 + s/c) \quad (81)$$

The coefficient matrix  $[\mathbf{A}^{II}]$  in equation (21) is equal to

$$\begin{bmatrix} 0 & \lambda/G \\ \lambda G & 0 \end{bmatrix} \quad (82)$$

## APPENDIX III

Propagator matrices  $[\mathbf{a}^I]$  and  $[\mathbf{a}^{II}]$

The elements of propagator matrix  $[\mathbf{a}^I(z)]$  in equation (26) are

$$\begin{aligned}
 (1, 1) &= (3, 3) = (1 - \rho_1)ch(\lambda z) + \rho_2 \lambda z sh(\lambda z) + \rho_1 ch(\lambda \rho z) \\
 (2, 2) &= (4, 4) = (1 + \rho_1)ch(\lambda z) - \rho_2 \lambda z sh(\lambda z) - \rho_1 ch(\lambda \rho z) \\
 (5, 5) &= (6, 6) = ch(\lambda \rho z) \\
 (1, 2) &= -(4, 3) = \rho_2 \lambda z ch(\lambda z) - \frac{1}{2} \left( \rho_3 - \frac{1-2\nu}{1-\nu} \right) sh(\lambda z) + \rho \rho_1 sh(\lambda \rho z) \\
 (1, 3) &= \frac{1}{2G} \left[ -\rho_2 \lambda z ch(\lambda z) + \frac{1}{2} \left( \rho_3 - \frac{3-4\nu}{1-\nu} \right) sh(\lambda z) - \rho \rho_1 sh(\lambda \rho z) \right] \\
 (1, 4) &= -(2, 3) = \frac{1}{2G} [\rho_1 ch(\lambda z) - \rho_1 ch(\lambda \rho z) - \rho_2 z sh(\lambda z)] \\
 (1, 5) &= \frac{(4, 5)}{2G} = \frac{(6, 2)}{2G \lambda s} = \frac{-(6, 3)}{\lambda s} = \frac{\rho_4}{\lambda} \left[ \frac{\rho_5 - 1}{\rho} sh(\lambda \rho z) - \rho_5 sh(\lambda z) \right] \\
 (1, 6) &= \frac{(4, 6)}{2G} = \frac{-(6, 4)}{\lambda^2 s \kappa} = \frac{-(2, 5)}{\lambda \kappa} = \frac{-(3, 5)}{2G \lambda \kappa} \\
 &= \frac{-(5, 3)}{\lambda s} = \frac{(5, 2)}{2G \lambda s} = \frac{(6, 1)}{2G \lambda^2 \kappa s} = \frac{\rho_4 \alpha_2}{\lambda} [ch(\lambda \rho z) - ch(\lambda z)] \\
 (2, 1) &= -(3, 4) = -\rho_2 \lambda z ch(\lambda z) + \frac{1}{2} \left( \rho_6 + \frac{1-2\nu}{1-\nu} \right) sh(\lambda z) - \frac{\rho_1}{\rho} sh(\lambda \rho z) \\
 (2, 4) &= \frac{1}{2G} \left[ \rho_2 \lambda z ch(\lambda z) - \frac{1}{2} \left( \rho_6 + \frac{3-4\nu}{1-\nu} \right) sh(\lambda z) + \frac{\rho_1}{\rho} sh(\lambda \rho z) \right] \\
 (2, 6) &= \frac{(3, 6)}{2G} = \frac{(5, 4)}{\lambda s} = \frac{-(5, 1)}{2G \lambda s} = \frac{\rho_4 \alpha_2}{\kappa} \left[ sh(\lambda z) - \frac{1}{\rho} sh(\lambda \rho z) \right] \\
 (3, 1) &= 2G \left[ -\rho_2 \lambda z ch(\lambda z) + (\rho_1 + \rho_2) sh(\lambda z) - \frac{\rho_1}{\rho} sh(\lambda \rho z) \right] \\
 (3, 2) &= -(4, 1) = 2G [-\rho_2 \lambda z sh(\lambda z) + \rho_1 ch(\lambda z) - \rho_1 ch(\lambda \rho z)] \\
 (4, 2) &= 2G \left[ \rho_2 \lambda z ch(\lambda z) - \frac{1}{2} \left( \rho_3 + \frac{1}{1-\nu} \right) sh(\lambda z) + \rho_1 \rho sh(\lambda \rho z) \right] \\
 (5, 6) &= \frac{(6, 5)}{a \kappa} = \frac{sh(\lambda \rho z)}{\lambda \rho \kappa}
 \end{aligned} \tag{83}$$

where

$$\begin{aligned}
 ch(x) &= \cosh(x), & sh(x) &= \sinh(x) \\
 \lambda\rho &= \sqrt{a/\kappa} = \sqrt{\lambda^2 + s/c} \\
 \alpha_2 &= \frac{1}{\lambda^2 - a/\kappa}, & \alpha_3 &= \frac{(1-2\nu)^2 b^2 s}{2G(1-\nu)^2 \kappa} \\
 \rho_1 &= \alpha_3 \alpha_2^2 \lambda^2, & \rho_2 &= \frac{1}{2} \left[ \alpha_2 \alpha_3 - \frac{1}{1-\nu} \right] \\
 \rho_3 &= \alpha_3 \alpha_2^2 (\lambda^2 + a/\kappa), & \rho_4 &= \frac{b(1-2\nu)}{2G(1-\nu)} \\
 \rho_5 &= \alpha_2 \lambda^2, & \rho_6 &= \alpha_3 \alpha_2^2 (3\lambda^2 - a/\kappa)
 \end{aligned} \tag{84}$$

The propagator matrix  $[\mathbf{a}^{\text{II}}(z)]$  is equal to

$$\begin{bmatrix} \cosh(\lambda z) & -\sinh(\lambda z)/G \\ -G \sinh(\lambda z) & \cosh(\lambda z) \end{bmatrix} \tag{85}$$

#### APPENDIX IV

*Solution matrices  $[\mathbf{Z}^{\text{I}}]$  and  $[\mathbf{Z}^{\text{II}}]$*

The elements of the solution matrix  $[\mathbf{Z}^{\text{I}}(z)]$  in equation (63) are

$$\begin{aligned}
 (1, 1) &= \left[ \frac{1}{G} - \lambda g(\lambda) \right] e^{\lambda y z} \\
 (2, 1) &= \lambda y g(\lambda) e^{\lambda y z} \\
 (3, 1) &= y e^{\lambda y z} \\
 (4, 1) &= e^{\lambda y z} \\
 (5, 1) &= 0 \\
 (6, 1) &= 0
 \end{aligned} \tag{86}$$

$$\begin{aligned}
 (1, 3) &= \left\{ -y \left[ \lambda \frac{dg(\lambda)}{d\lambda} + g(\lambda) \right] + \left[ \frac{1}{G} - \lambda g(\lambda) \right] z \right\} e^{\lambda y z} \\
 (2, 3) &= \lambda \left[ \frac{dg(\lambda)}{d\lambda} + g(\lambda) y z \right] e^{\lambda y z} \\
 (3, 3) &= -\left( \frac{1}{\lambda} - y z \right) e^{\lambda y z} \\
 (4, 3) &= z e^{\lambda y z}
 \end{aligned} \tag{87}$$

$$\begin{aligned}
(5, 3) &= \frac{df(\lambda)}{d\lambda} e^{\lambda yz} \\
(6, 3) &= -\lambda \kappa y \frac{dg(\lambda)}{d\lambda} e^{\lambda yz} \\
(1, 5) &= \left[ \frac{1}{G} - \lambda \rho g(\lambda \rho) \right] e^{\lambda \rho yz} \\
(2, 5) &= \lambda y g(\lambda \rho) e^{\lambda \rho yz} \\
(3, 5) &= y e^{\lambda \rho yz} / \rho \\
(4, 5) &= e^{\lambda \rho yz} \\
(5, 5) &= y f(\lambda \rho) e^{\lambda \rho yz} \\
(6, 5) &= -\lambda \kappa \rho f(\lambda \rho) e^{\lambda \rho yz}
\end{aligned} \tag{88}$$

where  $y = 1$ , and

$$\begin{aligned}
g(x) &= b(x)/d(x), \quad f(x) = c(x)/d(x) \\
b(x) &= \frac{1}{x} \left[ (\kappa x^2 - a) \left( x^2 + \frac{\lambda^2 v}{1-v} \right) - \frac{\lambda^2 b^2 s (1-2v)^2}{2G(1-v)^2} \right] \\
c(x) &= \frac{\lambda^2 b s (1-2v)(\lambda^2 - x^2)}{x(1-v)} \\
d(x) &= \frac{2G\lambda^2(\kappa x^2 - a)}{1-v} - \frac{\lambda^2 b^2 s (1-2v)^2}{(1-v)^2}
\end{aligned} \tag{89}$$

The elements  $(i, 2)$ ,  $(i, 4)$  and  $(i, 6)$  of  $[Z^I(z)]$  are obtained, respectively, from  $(i, 1)$ ,  $(i, 3)$  and  $(i, 5)$  by replacing  $y$  with  $-1$  ( $i = 1-6$ ).

The solution matrix  $[Z^II(z)]$  in equation (63) is equal to

$$\begin{bmatrix} e^{\lambda z} & e^{-\lambda z} \\ G e^{\lambda z} & -G e^{-\lambda z} \end{bmatrix} \tag{90}$$

## APPENDIX V

*Other quantities in terms of the cylindrical and Cartesian systems of vector functions*

In terms of the Cartesian system of vector functions, the horizontal discharges are expressed as

$$\begin{aligned}
q_x(x, y, z; s) &= i\kappa \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\alpha P - (\alpha f_M + \beta f_N)] S(x, y; \alpha, \beta) d\alpha d\beta \\
q_y(x, y, z; s) &= i\kappa \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\beta P - (\beta f_M + \alpha f_N)] S(x, y; \alpha, \beta) d\alpha d\beta
\end{aligned} \tag{91}$$

where  $f_M$  and  $f_N$  are the known expansion coefficients for a given fluid body-force  $f_i(x, y, z; s)$ , i.e.

$$\mathbf{f}(x, y, z; s) = \int_{-\infty}^{+\infty} [f_L(z)\mathbf{L}(x, y) + f_M(z)\mathbf{M}(x, y) + f_N(z)\mathbf{N}(x, y)] d\alpha d\beta \quad (92)$$

The horizontal stress components are given by

$$\begin{aligned} \sigma_{xx}(x, y, z; s) &= \int_{-\infty}^{+\infty} \left[ \frac{v}{1-v} T_L - 2G\alpha\beta U_N \right. \\ &\quad \left. + \frac{2G}{1-2v} \left( \frac{\lambda^2 v^2}{1-v} - (1-v)\alpha^2 - v\beta^2 \right) U_M - bP \right] S(x, y; \alpha, \beta) d\alpha d\beta \\ \sigma_{xy}(x, y, z; s) &= G \int_{-\infty}^{+\infty} [(\alpha^2 - \beta^2) U_N - 2\alpha\beta U_M] S(x, y; \alpha, \beta) d\alpha d\beta \\ \sigma_{yy}(x, y, z; s) &= \int_{-\infty}^{+\infty} \left[ \frac{v}{1-v} T_L + 2G\alpha\beta U_N \right. \\ &\quad \left. + \frac{2G}{1-2v} \left( \frac{\lambda^2 v^2}{1-v} - (1-v)\beta^2 - v\alpha^2 \right) U_M - bP \right] S(x, y; \alpha, \beta) d\alpha d\beta \end{aligned} \quad (93)$$

Similarly, in terms of the cylindrical system of vector functions, the horizontal discharges can be expressed as

$$\begin{aligned} q_r(r, \theta, z; s) &= -\kappa \sum_m \int_0^{+\infty} \left[ p \frac{\partial}{\partial r} - \left( f_M \frac{\partial}{\partial r} + f_N \frac{\partial}{r \partial \theta} \right) \right] S(r, \theta; \lambda, m) \lambda d\lambda \\ q_\theta(r, \theta, z; s) &= -\kappa \sum_m \int_0^{+\infty} \left[ p \frac{\partial}{r \partial \theta} - \left( f_M \frac{\partial}{r \partial \theta} - f_N \frac{\partial}{\partial r} \right) \right] S(r, \theta; \lambda, m) \lambda d\lambda \end{aligned} \quad (94)$$

where  $f_M$  and  $f_N$  are the expansion coefficients for the given fluid body-force  $f_i(x, y, z; s)$ , i.e.

$$\mathbf{f}(r, \theta, z; s) = \sum_m \int_0^{+\infty} [f_L(z)\mathbf{L}(r, \theta) + f_M(z)\mathbf{M}(r, \theta) + f_N(z)\mathbf{N}(r, \theta)] \lambda d\lambda \quad (95)$$

The horizontal stress components are derived as

$$\begin{aligned} \sigma_{rr}(r, \theta, z; s) &= \sum_m \int_0^{+\infty} \left[ \frac{v}{1-v} T_L + 2GU_N \Delta_1 \right. \\ &\quad \left. - 2G(\lambda^2 + \Delta_2) U_M - bP \right] S(r, \theta; \lambda, m) \lambda d\lambda \\ \sigma_{r\theta}(r, \theta, z; s) &= G \sum_m \int_0^{+\infty} [(\lambda^2 + 2\Delta_2) U_N + 2\Delta_1 U_M] S(r, \theta; \lambda, m) \lambda d\lambda \\ \sigma_{\theta\theta}(r, \theta, z; s) &= \sum_m \int_0^{+\infty} \left[ \frac{v}{1-v} T_L - 2GU_N \Delta_1 \right. \\ &\quad \left. - 2G(\lambda^2 - \Delta_2) U_M - bP \right] S(r, \theta; \lambda, m) \lambda d\lambda \end{aligned} \quad (96)$$

where  $\Delta_1$  and  $\Delta_2$  are two surface operators defined as

$$\Delta_1 = \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial}{\partial \theta}, \quad \Delta_2 = \frac{1}{r^2} \frac{\partial}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad (97)$$

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